6/H-29 (vii) (Syllabus-2015)

2018

(April)

MATHEMATICS

(Honours)

(Advanced Calculus)

(GHS-61)

Marks : 75

Time : 3 hours

The figures in the margin indicate full marks for the questions

Answer **five** questions, selecting **one** from each Unit

UNIT—I

1. *(a)* If a function f is bounded and integrable on $[a, b]$ and there exists a function F such that $F'(x) = f(x) \ \forall x \in [a, b]$, then show that

$$\int_a^b f(x)dx = F(b) - F(a)$$ 6

(Turn Over)

(b) Show that a bounded function f having only a finite number of points of discontinuity on $[a, b]$ is integrable on $[a, b]$.

(c) Show that

$$\left|\int_p^q \frac{\sin x}{x}\,dx\right| \le \frac{2}{p}$$

if $q > p > 0$.

2. (a) Show that

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

is convergent iff both m and n are positive.

(b) Let ϕ be bounded and monotonic on $[a, \infty[$ and $\int_a^\infty f(x)dx$ be convergent. Show that $\int_a^\infty f(x)\,\phi(x)\,dx$ is convergent.

(c) Show that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx = \log\left(\frac{b}{a}\right)$$

(Continued)

UNIT—II

3. (a) Let f be a continuous function on $[a, b]\times[c, d]$ and let $\phi(y) = \int_a^b f(x, y)\,dx$. If f_y exists and is continuous, show that ϕ is differentiable and $\phi'(y) = \int_a^b f_y(x, y)\,dx$. 8

(b) Find the value of

$$\int_0^\pi \frac{dx}{a + b\cos x}$$

where $a > 0$ and $|b| < a$. 7

4. (a) If $f(x, y)$ is continuous where $c \le y \le d$ and $a \le x$ and the integral $\phi(y) = \int_a^\infty f(x, y)dx$ is uniformly convergent, show that ϕ can be integrated under the integral sign. 7

(b) Establish the right to integrate

$$\int_0^\infty e^{-xy}\cos mx\,dx$$

under the integral sign and deduce that

$$\int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\cos mx\,dx = \frac{1}{2}\log\frac{b^2 + m^2}{a^2 + m^2}$$

where $a > 0$, $b > 0$. 4+4=8

(Turn Over)

UNIT—III

5. (a) Show that

$$\int_a^b \int_{a^2/x}^{x} F dx dy = \int_{a^2/b}^{a} \int_{a^2/y}^{b} F dx dy + \int_a^b \int_y^b F dx dy$$

(b) Evaluate $\iint x^2 y^2 dx dy$ over the domain $\{(x, y) : x \geq 0,\ y \geq 0 \text{ and } x^2 + y^2 \leq 1\}$.

(c) State Green's theorem. Verify Green's theorem by evaluating in two ways the integral $\int (x^2 y dx + x y^2 dy)$ taken along the closed path formed by $y = x$, $x^2 = y^3$ in the first quadrant.

6. (a) Evaluate $\int_C (x^2 + y^2) dx$ and $\int_C (x^2 + y^2) dy$ where C is the arc of the parabola $y^2 = 4ax$ between (0, 0) and $(a, 2a)$.

(b) Show that

$$\int_0^1 \left\{ \int_0^1 \frac{x - y}{(x + y)^3} dy \right\} dx = \frac{1}{2} \text{ and}$$

$$\int_0^1 \left\{ \int_0^1 \frac{x - y}{(x + y)^3} dx \right\} dy = -\frac{1}{2}$$

(Continued)

UNIT—IV

7. (a) Define interior point, open set and limit point in $\mathbb{R}^n$. 1+1+1=3

(b) Give examples of the following with brief justification : 1+1=2

(i) An infinite bounded set with two limit points

(ii) A bounded set which is neither closed nor open

(c) State and prove Cantor's intersection theorem. 6

(d) Prove that an arbitrary union of open sets in $\mathbb{R}^n$ is open. 4

8. (a) Let $S, T \subseteq \mathbb{R}^n$. Show that

(i) $(S \cup T)' = S' \cup T'$ 4+4=8

(ii) $\text{int}(S \cap T) = \text{int}(S) \cap \text{int}(T)$

(b) Find the interior and the set of limit points of the set $\left\{ \frac{1}{n} + \frac{1}{m} : m, n \in \mathbb{N} \right\}$. 2

(c) If f is continuous on $X \subseteq \mathbb{R}^n$ and A is a compact subset of X, show that $f(A)$ is compact. 5

(Turn Over)

UNIT—V

9. (a) Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $(0, 1]$. 4

(b) If f is continuous and strictly increasing on $[a, b]$, show that f^{-1} is also continuous and strictly increasing on $[f(a), f(b)]$. 5

(c) Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) \cdot f(b) < 0$, show that there is a point c between a and b such that $f(c) = 0$. 6

10. (a) (i) Define partial derivative and directional derivative of a real-valued function f defined on $\mathbb{R}^2$ at a point (a, b).

(ii) Show that a function f defined by $f(x, y) = \frac{x^3 + y^3}{x - y}$ if $x \neq y$, $f(x, y) = 0$ if $x = y$ is not continuous at the origin but the first order partial derivatives exist at that point. 2+4=6

(b) Prove that a real-valued function f of two variables is differentiable at a point (a, b) if it has continuous first-order partial derivatives at that point. 5

(c) If

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

where $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$, show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. 4

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